

The Generalized Shiryaev's Problem and Skorohod Embedding[☆]

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Abstract

In this paper we consider a connection between the famous Skorohod embedding problem and the Shiryaev inverse problem for the first hitting time distribution of a Brownian motion: given a probability distribution, F , find a boundary such that the first hitting time distribution is F . By randomizing the initial state of the process we show that the inverse problem becomes analytically tractable. The randomization of the initial state allows us to significantly extend the class of target distributions in the case of a linear boundary and moreover allows us to establish connection with the Skorohod embedding problem.

Key words: First hitting time, Shiryaev's inverse boundary problem, Skorohod embedding

1. Introduction

Let X_t be an arbitrary process with the left-continuous sample paths and let $b(t)$ be a continuous absorbing boundary satisfying the condition $X_0 \geq b(0)$.

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The random variable

$$\tau = \begin{cases} \inf\{t \geq 0 : X_t < b(t)\}, & \text{if there exists } t \text{ such that } X_t < b(t), \\ \infty, & \text{otherwise,} \end{cases}$$

is called the first hitting time for the process X_t . The problem

$$\text{Given a process, } X_t, \text{ and the boundary, } b(t), \quad (1)$$

$$\text{find the distribution } F_\tau(t) = \mathbb{P}(\tau \leq t).$$

is the starting point of a very rich research area in the theory of stochastic processes. If X_t is a diffusion process, the problem of finding the distribution of τ is a classical one. The first papers on the problem were published by P. Levy, A. Khintchine and A. Kolmogorov in the 1920s. These results are discussed in the monograph² Khintchine (1933). Khintchine gave a complete solution of the problem for sufficiently smooth boundaries. This solution was expressed in terms of a boundary value problem for the associated partial differential operator (infinitesimal generator). Since then many books and research papers were published in this area. The monograph Lerche (1986) summarizes known analytical results obtained by the mid-1980s. In Karatzas and Shreve (2005) the link between analytical methods and the martingale approach is considered, the paper Durbin (1971) discusses computational aspects of the problem, while the Taylor expansions of the probability distribution of τ are considered in Hobson et al. (1999).

The following, inverse to (1), problem was proposed by Albert Shiryaev in

²Khintchine called (1) the second problem of diffusion.

the mid-70's in his Banach Center lectures:

$$\begin{aligned} & \textit{Given a process, } X_t, \textit{ and a distribution, } F(t), \\ & \textit{find a boundary, } b(t), \textit{ such that } \mathbb{P}(\tau \leq t) = F(t). \end{aligned} \tag{2}$$

Problem (2), to the best of our knowledge, was proposed by A. Shiryaev in the case when X_t is a Brownian motion and $F(t)$ is an exponential distribution. If X_t is a Brownian motion with a random initial value, i.e., the process $X_t = \xi + W_t$ is a Brownian motion starting at a random point ξ , we will refer to Problem (2) as a generalized Shiryaev's problem (GSP).

The third problem considered in this paper is the Skorohod embedding problem:

$$\begin{aligned} & \textit{Given a probability measure, } \mu, \textit{ with a finite second moment,} \\ & \textit{and a Wiener process, } W_t, \textit{ find an integrable stopping time, } \tau_*, \\ & \textit{such that the distribution of } W_{\tau_*} \textit{ is } \mu. \end{aligned} \tag{3}$$

This problem has a long list of references. Here we mention only the original paper Skorohod (1965) and a survey by Obloj (2004) where many important applications and various solutions of the Skorohod embedding problem can be found.

It is intuitively appealing that there must be a connection between the problems (2) and (3). Indeed if a monotone continuous function $b_*(\cdot)$ solves Problem (2) for a Wiener process W_t , and the distribution function $F(t)$, then the first hitting time, τ_* , is a solution of the Skorohod problem with $\mathbb{P}(W_{\tau_*} \leq t) = F(b_*^{-1}(t))$. The only problem with the latter condition is that $b_*(\cdot)$ is usually unknown in analytical form. Moreover, the boundary might not belong to the class of monotone functions.

Models using the first hitting time distribution find extensive application in

the areas of portfolio credit risk modeling (see Iscoe and Kreinin (1999), Iscoe et al. (1999), Schmidt and Novikov (2008)) and pricing of credit derivatives (Avellaneda and Zhu (2001), Hull and White (2001)). In these contexts the process represents the so-called *distance to default* of an obligor (see Avellaneda and Zhu (2001)), while the first hitting time represents a default event. The boundary therefore acts a barrier separating the healthy states of the obligor from the default state. For this reason, the boundary, $b(t)$, is often called the *default boundary* in the applied literature on credit risk modeling. In particular, an interesting model of default events with a randomized boundary was proposed in Schmidt and Novikov (2008).

The inverse problem (2) was considered in Iscoe et al. (1999) when the process X_t is a Brownian random walk³. A detailed analysis of the inverse problem in the discrete time setting is given in Iscoe and Kreinin (1999) as well as a Monte Carlo based solution. Their approach is applicable to a much more general class of processes X_t , not just Brownian random walks, and is computationally simple to implement.

Existence of the solution to the continuous-time inverse problem (2) is analyzed in Chadam et al. (2006). In Peskir (2002), an integral equation for the boundary is derived when X_t is a Brownian motion. A general analysis of the integral equations for the boundary and existence and uniqueness theorems are considered in Jaimungal et al. (2009a). The randomized inverse problem is considered in Jaimungal et al. (2009b) and in the preliminary publication Jackson et al. (2009).

The Shiryaev problem is notoriously difficult and analytical solutions are known only in a few cases only (see Shepp (1967), Salminen (1988), Lerche

³A Brownian random walk is a discrete time process with Gaussian increments and variance proportional to the time step.

(1986), Peskir and Shiryaev (2006), Breiman (1967), Alili and Patie (2005), Novikov (1981), Peskir (2002)). However, existence of the solution to the problem has been proven for an arbitrary target distribution, F , by Dudley and Gutmann (1977) and by Anulova (1980). Unfortunately, an analog of this existence theorem in its most general form cannot be proven for the *the Generalized Shiryaev's problem* and in fact counter-examples do exist (see Jaimungal et al. (2009b) and Proposition 2 below). Nonetheless, the randomized version does admit closed form solutions for a large class of distributions of the first hitting time, F , including the gamma distribution (see Jackson et al. (2009) and more generally Jaimungal et al. (2009b)) as well as a subset of one-sided stable distributions (see Corollary 1 below).

One of our main goals is to establish a connection between the Generalized Shiryaev's problem and the Skorohod embedding problem. The remainder of this paper is organized as follows. In Section 2 we proved a new short derivation for the Laplace transform of the distribution of the first hitting time and the distribution of the initial random position ξ of the process X_t . When the boundary is linear, we find solutions for a class of mixtures of gamma distributions of the first hitting time and for a class of stable distributions. In Section 3 we analyze the structure of the solutions to the Shiryaev's problem and introduce the minimal solution which possess a very elegant structure. The randomization of the initial state of the process allows us to stretch the boundary and transform it into a straight line. In this case we have a simple relation between the distributions of τ and X_τ . This observation allows us to connect the Skorohod embedding problem to the Generalized Shiryaev problem as shown in Section 4.

This paper is self-contained; it represents an extended version of the talk given at the 5th Bachelier colloquium, January 2011, Metabief.

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2. Randomization of the initial state

2.1. General equation

Let $(\Omega, \mathfrak{B}, \mathfrak{F}_t, \mathbb{P})$ be a filtered probability space and let $\Omega = \mathbf{C}([0, \infty)) \times \mathbb{R}^1$ be the Cartesian product of the space of continuous functions on the positive semi-axis and \mathbb{R}^1 . Consider a random variable ξ on this probability space. We assume that $\{\mathfrak{F}_t\} = \{\mathfrak{F}_t^W\} \vee \mathcal{B}(\mathbb{R}^1)$ is an augmented filtration where \mathfrak{F}_t^W is the natural filtration, generated by the standard Wiener process W_t and assume that $\mathfrak{B} = \bigcup_{t \geq 0} \mathfrak{F}_t$. We also assume that the random variable ξ is independent of the process W_t .

Let $X_t = \xi + W_t$ and let $\mathbb{P}(0 \leq \xi < +\infty) = 1$. Consider now the linear boundary, $b(t) = k t$, $k \geq 0$. Denote the first hitting time of the process X_t to $b(t)$ by $\tau^{(k)}$, i.e.,

$$\tau^{(k)} = \inf\{t : t > 0, \xi + W_t < k t\}.$$

The Generalized Shiryaev's Problem (GSP) for the process X_t is formulated as follows:

$$\begin{aligned} & \text{Given a distribution } F(t), \text{ find } \xi, \\ & \text{such that } \mathbb{P}(\tau^{(k)} \leq t) = F(t). \end{aligned} \tag{4}$$

The triplet $(\xi, \tau^{(k)}, k)$ is called a solution of the Generalized Shiryaev's Problem.

We are interested in the solutions to the GSP and their properties. In

particular, we are interested in the solution of Problem (4) when $F(t)$ belongs to the class of Gamma distributions, $F \in \Gamma_{\lambda, \gamma}$, with the probability density function

$$p_\gamma(t) = \lambda \cdot \frac{(\lambda t)^{\gamma-1}}{\Gamma(\gamma)} \cdot \exp(-\lambda t), \quad \gamma > 0, \lambda > 0.$$

Remark 1. The case $\gamma = 1$ corresponds to the original formulation of Shiryaev's problem with a randomized initial point. In this case the distribution $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$.

To solve the problem, we will derive a connection between the Laplace transforms of the target distribution $F(t)$ and the initial starting point ξ . To this end, denote

$$\hat{f}(s) = \mathbb{E}[e^{-s\tau}] = \int_0^\infty e^{-st} dF(t), \quad s \in \mathbb{R}_+, \quad (5)$$

where, as usual, $\mathbb{R}_+ = \{s : s \geq 0\}$, is the set of non-negative real numbers, and let $\hat{g}(s) = \mathbb{E}[e^{-s\xi}]$. The function $\hat{f}(s)$ on $[0, \infty)$, is completely monotone (see Feller (1971)), $\hat{f} \in \mathfrak{M}$, where

$$\mathfrak{M} = \left\{ V(s) : (-1)^n \frac{d^n V(s)}{ds^n} \geq 0, \quad n = 0, 1, \dots, \quad s \in \mathbb{R}_+ \right\}.$$

The class of completely monotone functions form an algebra: the sum and the product of completely monotone functions belong to \mathfrak{M} . According to the classical Bernstein's Theorem (see Feller (1971)), if $\hat{f}(s) \in \mathfrak{M}$ and $\hat{f}(0) = 1$, then there exists a cumulative distribution function, $F(t)$, satisfying (5).

The following statement for completely monotone functions is very well known (see Feller (1971), Criterion 2):

Proposition 1. *If $G(s)$ is a completely monotone function, $u(s) \geq 0$ and the first derivative $u' \in \mathfrak{M}$ then $G(u(s)) \in \mathfrak{M}$.*

Let us now derive the main equation for the Laplace transforms of the distributions of τ and ξ .

Theorem 1. *If a random variable ξ is a solution to the Problem (4) then the function $\hat{g}(s)$ satisfies the equation*

$$\hat{g}(s) = \hat{f}(sk + s^2/2). \quad (6)$$

Proof. The process X_t is a martingale with respect to the filtration $\{\mathfrak{F}_t\}_{t \geq 0}$. Consider the exponential martingale $M_t = \exp(-sX_t - s^2t/2)$, ($s > 0$). It is not difficult to show that if ξ is finite almost surely then the exponential martingale is bounded for all $t \leq \tau$. Therefore, by the optional stopping theorem,

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

Moreover, the expected value

$$\mathbb{E}[M_0] = \mathbb{E}[e^{-s\xi}].$$

Furthermore, he have $X_\tau = k\tau$ and

$$M_\tau = \exp(-\tau \cdot (sk + s^2/2)).$$

Therefore $\mathbb{E}[M_\tau] = \hat{f}(ks + s^2/2)$. Finally, we obtain Equation (6). ■

2.2. Gamma-distributed first hitting time

In the case $f \in \Gamma_{\lambda, \gamma}$, the random variable ξ solving Problem (4) admits the following simple probabilistic interpretation.

Theorem 2. *Let $f(t) = p_\gamma(t)$, for some $\gamma > 0$. Suppose that $2\lambda \leq k^2$. Then $\hat{g}(s)$ satisfies*

$$\hat{g}(s) = \frac{(k + \sqrt{k^2 - 2\lambda})^\gamma}{(s + k + \sqrt{k^2 - 2\lambda})^\gamma} \cdot \frac{(k - \sqrt{k^2 - 2\lambda})^\gamma}{(s + k - \sqrt{k^2 - 2\lambda})^\gamma}. \quad (7)$$

and the random variable

$$\xi = \xi_1 + \xi_2, \quad (8)$$

where ξ_1 and ξ_2 are independent gamma-distributed random variables with a common shape parameter γ : $\xi_1 \sim \Gamma(\gamma, \lambda_1)$ and $\xi_2 \sim \Gamma(\gamma, \lambda_2)$, with

$$\lambda_1 = k - \sqrt{k^2 - 2\lambda}, \quad \lambda_2 = k + \sqrt{k^2 - 2\lambda}. \quad (9)$$

Proof. We have

$$\hat{f}(s) = \frac{\lambda^\gamma}{(\lambda + s)^\gamma}.$$

From (6) we obtain

$$\hat{g}(s) = \left(\frac{2\lambda}{s^2 + 2ks + 2\lambda} \right)^\gamma.$$

If $k^2 \geq 2\lambda$ the quadratic equation

$$s^2 + 2ks + 2\lambda = 0$$

has real roots $-\lambda_1$ and $-\lambda_2$. Then, taking into account that $\lambda_1\lambda_2 = 2\lambda$, we find

$$\hat{g}(s) = \left(\frac{\lambda_1}{\lambda_1 + s} \right)^\gamma \cdot \left(\frac{\lambda_2}{\lambda_2 + s} \right)^\gamma.$$

The additive representation (8) for ξ follows immediately from the latter equation. ■

Remark 2. It follows from Theorem 2 that if the GSP has a solution for $k = k_*$, given $f(t) \in \Gamma_{\lambda, \gamma}$, then it also has a solution for any $k > k_*$. We will generalize this property for a larger class of distributions in the next section.

Theorem 2 can also be generalized in the following direction. Consider a class of random variables \mathcal{M}_Γ which are mixtures of gamma-distributed random variables with respect to shape γ and scale λ parameters. Write,

$$\tau = \text{Mix}(\Gamma_{\lambda, \gamma}, \mu),$$

where μ is a mixing measure having a support, $(\lambda, \gamma) \in S = [0, \lambda_*] \times [0, \infty)$, $\mu(S) = 1$, and $\lambda_* = k^2/2$. Then

$$\hat{f}(s) := \mathbb{E}[e^{-s\tau}] = \int_S \frac{\lambda^\gamma}{(\lambda + s)^\gamma} d\mu(\lambda, \gamma).$$

In this case

$$\hat{g}(s) = \int_S \frac{\lambda_1^\gamma(\lambda)}{(\lambda_1(\lambda) + s)^\gamma} \frac{\lambda_2^\gamma(\lambda)}{(\lambda_2(\lambda) + s)^\gamma} d\mu(\lambda, \gamma), \quad (10)$$

where $\lambda_1(\lambda)$ and $\lambda_2(\lambda)$ satisfy (9). Therefore, we see that the random variable ξ which leads to the hitting time distribution $\text{Mix}(\Gamma_{\lambda, \gamma}, \mu)$ is in fact a mixture of the sums of independent gamma-distributed random variables.

This generalization is interesting because it is known that the class of finite mixtures of gamma-distributed random variables $\eta \in \mathcal{M}_\Gamma$ form a dense set in the space of random variables. Moreover, an arbitrary random variable η_* can

be obtained as a weak limit as the number of gamma distributions $n \rightarrow \infty$,

$$\eta_n \xrightarrow{w} \eta_*, \quad \eta_n \in \mathcal{M}_\Gamma.$$

Unfortunately, this sequence of approximations can not be used to extend the solution (10) through weak convergence because the parameter λ has a bounded support. In particular, we cannot approximate constant almost surely random variables using the random variables, $\eta_n \in \mathcal{M}_\Gamma$.

2.3. Stable distribution of the first hitting time

So far, we have considered the case when τ has finite mean, $\mathbb{E}\tau < +\infty$. However, it is also interesting to analyze the case of one-sided stable distributions with infinite expected value. For this case, we take the slope coefficient $k = 0$. Then, if one takes the random variable τ such that

$$\mathbb{E}[e^{-s\tau}] = e^{-c s^\alpha}, \quad c > 0, \quad 0 < \alpha < 1,$$

then it follows from (6) that

$$\hat{g}(s) = \exp(-c_1 s^\beta),$$

where $\beta = 2\alpha$ and $c_1 = \frac{c}{2\alpha}$. Thus, we obtain

Corollary 1. *If τ has a stable distribution with the parameter α , $0 < \alpha < 1/2$, then ξ has a stable distribution with the parameter 2α .*

3. Structure of the solutions to Shiryaev's problem

3.1. General results

Theorem 2 describes the solutions to the Shiryaev's problem when the first hitting time τ is gamma-distributed. In this case, it can be shown that the condition $k \geq k_* = \sqrt{2\lambda}$ is necessary and sufficient for existence of the random variable ξ (see Jackson et al. (2009) and Jaimungal et al. (2009b)). Thus if the problem has a solution for the slope k_* then it has a solution for any $k > k_*$. The following statement generalizes this result.

Theorem 3. *Given τ , suppose there exist $k_* > 0$ and $\xi_* = \xi(k_*)$ such that Equation (6) is satisfied:*

$$\hat{g}_*(s) := \mathbb{E}[e^{-s\xi_*}] = \hat{f}(k_*s + s^2/2).$$

Then for any $k > k_*$ there exists a random variable $\xi(k)$ such that

$$\mathbb{E}[e^{-s\xi(k)}] = \hat{f}(ks + s^2/2), \quad s \geq 0. \quad (11)$$

Proof. The function $\hat{g}(s) = \hat{f}(ks + s^2/2)$ satisfies the relation $\hat{g}(0) = 1$. Therefore, it is enough to prove that $\hat{g} \in \mathfrak{M}$. Let $k > k_*$. Consider the function

$$u(s) = \sqrt{s^2 + 2ks + k_*^2} - k_*.$$

Obviously, $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and

$$\frac{s^2}{2} + ks = \frac{u^2(s)}{2} + k_*u(s), \quad s \geq 0.$$

The first derivative of $u(s)$ is

$$\frac{d}{ds}u(s) \equiv u'(s) = (s + k) \cdot (s^2 + 2ks + k_*^2)^{-\frac{1}{2}}.$$

Let us prove that the function $u'(s) \in \mathfrak{M}$. We have

$$u'(s) \cdot (u(s) + k_*) = s + k. \quad (12)$$

Then if $k > k_*$, we derive from (12) the inequality

$$u'(s) > 1. \quad (13)$$

Denote n^{th} derivative of the function $u(s)$ by $u^{(n)}(s)$. ($n = 1, 2, \dots$). Differentiating Equation (12) we find

$$u^{(2)}(s) \cdot (u(s) + k_*) + (u'(s))^2 = 1. \quad (14)$$

The latter Equation and (13) imply the inequality

$$u^{(2)}(s) < 0, \quad s > 0.$$

Denote $m = \lceil n/2 \rceil$. From Equation (14) we derive

$$u^{(n)}(s) \cdot (u(s) + k_*) = - \sum_{j=1}^m C_{n,j} u^{(j)}(s) \cdot u^{(n-j)}(s), \quad (15)$$

The coefficients $C_{n,j}$ satisfy the relations

$$\begin{aligned} C_{n+1,1} &= C_{n,1} + 1, \quad n = 2, 3, \dots, \\ C_{n+1,j} &= C_{n,j} + C_{n,j-1}, \quad n = 2, 3, \dots, j = 2, \dots, m-1. \end{aligned}$$

Taking into account that $C_{2,1} = 1$, we find

$$C_{n,j} = \binom{n}{j}, \quad j = 1, 2, \dots, m-1. \quad (16)$$

The coefficient $C_{n,m}$ satisfies the relation

$$C_{n,m} = \begin{cases} \binom{n}{m}, & \text{if } n \text{ is odd,} \\ \frac{1}{2} \binom{n}{m}, & \text{otherwise,} \end{cases} \quad (17)$$

Thus, $C_{n,j} > 0$ in Equation (15). Then from (15) we prove by induction that

$$(-1)^{n+1} u^{(n)}(s) > 0 \quad \text{for all } n = 1, 2, \dots$$

Therefore $u'(s) \in \mathfrak{M}$. Then we have

$$\hat{g}(s) = \hat{f} \left(\frac{u^2(s)}{2} + k_* u(s) \right) = \hat{g}_*(u(s)).$$

Since $\hat{g}_* \in \mathfrak{M}$ and $u'(s) \in \mathfrak{M}$ we derive from Proposition 1 that $\hat{g}(s) \in \mathfrak{M}$, as was to be proved. ■

The next proposition demonstrates that there exist parameters of the GSP such that the problem does not have a solution.

Proposition 2. *Suppose $\mathbb{E}[\tau^2] < \infty$. Then the Problem (4) does not have a solution if $k < \sqrt{\mathbb{E}[\tau]}/\sigma(\tau)$.*

Proof. Indeed, from Equation (6) we find

$$\frac{d\hat{g}(s)}{ds} = \frac{d\hat{f}(ks + s^2/2)}{ds} \cdot (k + s),$$

and

$$\frac{d^2\hat{g}(s)}{ds^2} = \frac{d^2\hat{f}(ks + s^2/2)}{ds^2} \cdot (k + s)^2 + \frac{d\hat{f}(ks + s^2/2)}{ds}.$$

Substituting $s = 0$ into these equations, we find

$$\begin{aligned} \mathbb{E}[\xi] &= k \cdot \mathbb{E}[\tau], \\ \mathbb{E}[\xi^2] &= k^2 \mathbb{E}[\tau^2] - \mathbb{E}[\tau]. \end{aligned}$$

The equations for the first two moments of ξ imply

$$k^2 \sigma^2(\tau) = \mathbb{E}[\tau] + \sigma^2(\xi) \geq \mathbb{E}[\tau].$$

Finally, we obtain

$$k \geq \frac{\sqrt{\mathbb{E}[\tau]}}{\sigma(\tau)}.$$

Proposition 2 is thus proved. ■

Remark 3. Proposition 2 provides only a necessary condition for existence of a solution to Problem (4). In particular, if $\tau \sim \Gamma(\gamma, \lambda)$, then the necessary and sufficient condition for existence is $k \geq \sqrt{2\lambda}$ and the minimal solution corresponds to $k_* = \sqrt{2\lambda}$.

Remark 4. If τ is a constant with probability 1, Equation (6) does not have a solution. The GSP can not be solved using our randomization approach in this case.

Theorem 3 and Proposition 2 imply that the admissible set for the coefficient k is either the semi-infinite interval $[k_*, \infty)$ or an empty set. If τ has finite first two moments and the random variable ξ exists for some $k_* > 0$ then for each $k > k_*$ there exists a random variable $\xi(k)$ solving Problem (4). In this manner, we can obtain a family of solutions $\Xi = \{\xi(k)\}$ parameterized by the slope coefficient k . Furthermore, the variance of the random variable $\xi(k)$ is a monotone function of k and the one with minimal variance is what we call the *minimal solution* to Problem (4). In particular, if τ has an exponential distribution with parameter λ , then the random variable ξ_* corresponds to the minimal solution has slope $k_* = \sqrt{2\lambda}$. In this case the random variable $\xi(k_*)$ is Erlang distributed with order two.

3.2. Esscher families

Consider a non-negative random variable η and denote $\hat{f}_\eta(s) = \mathbb{E}[e^{-s\eta}]$, $s \geq 0$ its Laplace transform. It is convenient to introduce a family of random variables,

$$\mathcal{E}(\eta) = \left\{ \eta_a : \mathbb{E}[e^{-s\eta_a}] = \hat{f}_\eta(a+s)/\hat{f}_\eta(a), \quad a > 0 \right\},$$

which we call the *Esscher family generated by the random variable η* . As before, let τ be the first hitting time and $\hat{f}(s) = \mathbb{E}[e^{-s\tau}]$ be its Laplace transform. Suppose the generalized Shiryaev's Problem (4) has a solution $\xi = \xi(k_0)$ for some fixed slope k_0 . Consider now two Esscher families $\mathcal{E}(\tau)$ and $\mathcal{E}(\xi)$ generated by τ and ξ , respectively.

Proposition 3. *For each $k > k_0$ there exist $\alpha = \alpha(k)$ and $\beta = \beta(k)$ such that $\tau_\alpha \in \mathcal{E}(\tau)$ and $\xi_\beta \in \mathcal{E}(\xi)$ and $(\tau_\alpha, \xi_\beta, k)$ is a solution to the Generalized Shiryaev's problem.*

Proof. Denote $\Delta = k - k_0$, ($\Delta > 0$), and take $\alpha = \Delta \cdot k_0 + \Delta^2/2$, $\beta = \Delta$. Consider the random variable $\xi_\beta \in \mathcal{E}(\xi)$. Then after simple transformations we find

$$\hat{g}_\beta(s) = \frac{\hat{f}(\alpha + (k_0 + \Delta) \cdot s + s^2/2)}{\hat{f}(\alpha)} = \frac{\hat{f}(\alpha + ks + s^2/2)}{\hat{f}(\alpha)} = \hat{f}_\alpha(ks + s^2).$$

Thus, the triplet $(\tau_\alpha, \xi_\beta, k)$ form a solution of the problem. ■

4. Connection to Skorohod problem.

We are now in a position to construct a solution to the Skorohod problem through the solution to the Generalized Shiryaev's problem. Suppose that the distribution of the stopped process X_τ has a non-negative support, $\mathbb{P}(X_\tau \geq 0) = 1$, and the continuous distribution, $F_X(x) = \mathbb{P}(X_\tau \leq x)$, can be matched by a corresponding distribution of the random variable ξ for some $k > 0$. Taking into account that $X_\tau = k\tau$ for the linear boundary $b(t) = kt$, ($k > 0$), we immediately obtain that $\mathbb{P}(\tau \leq t) = F_X(kt)$ and the Laplace transform of the density of τ satisfies

$$\hat{f}_\tau(s) = \hat{f}_X\left(\frac{s}{k}\right).$$

In this case

$$\hat{g}(s) = \hat{f}_X\left(\frac{s^2}{2k^2} + s\right).$$

Similarly, the Skorohod problem can be solved for the distributions with $\mathbb{P}(X_\tau < 0) = 1$. In this case, the first hitting time is understood as

$$\tau = \inf_{t>0} \{t : X_t > k t, \quad k < 0\},$$

i.e., the first time the process touches the boundary from below. Clearly, the initial point ξ must then have support on the negative real axis.

The solution to the general Skorohod problem – without restrictions on the support of the distribution of X_τ – can then be obtained as follows: Let us represent the random variable X_τ as a mixture of the random variables $X_+ > 0$ and $X_- < 0$:

$$X_\tau = \begin{cases} X_+, & \text{with probability } p_+ = \mathbb{P}(X_\tau \geq 0), \\ X_-, & \text{with probability } p_- = 1 - p_+. \end{cases}$$

Suppose that the Skorohod problem for the random variables X_+ and X_- can be solved by the boundaries $b_+(t) = k_+ \cdot t$ and $b_-(t) = k_- \cdot t$ and the random variables ξ_+ and ξ_- , respectively (see Figure 1). Then we have the following

Proposition 4. *The random variable*

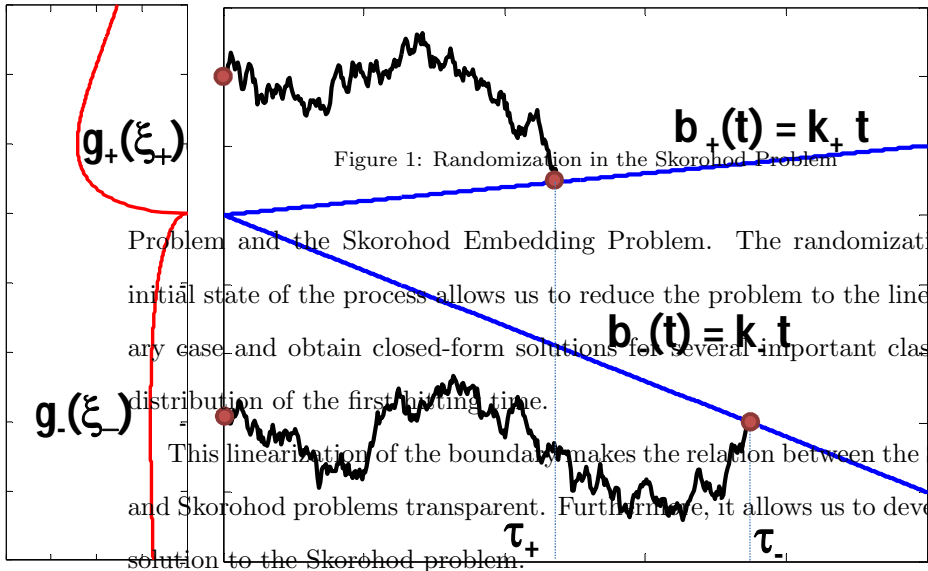
$$\xi = \begin{cases} \xi_+, & \text{with probability } p_+, \\ \xi_-, & \text{with probability } p_-, \end{cases}$$

and the boundary $b(t) = b_+ \cup b_-$ solve the Skorohod problem for the random variable X_τ .

Remark 5. In this manner, the Skorohod problem is solved by randomizing the starting point of the Brownian motion over the entire real line, and searching for the first hitting time when the process enters the wedge region described by the lines $k_+ t$ and $k_- t$.

5. Conclusion

We have demonstrated that randomization of the initial state of the process is a very powerful tool for solving and tying together the Generalized Shiryaev's



Problem and the Skorohod Embedding Problem. The randomization of the initial state of the process allows us to reduce the problem to the linear boundary case and obtain closed-form solutions for several important classes of the distribution of the first hitting time.

This linearization of the boundary makes the relation between the Shiryaev's and Skorohod problems transparent. Furthermore, it allows us to develop a new solution to the Skorohod problem.

We close this paper with a brief discussion of the directions for future research. In the present paper we discussed the case of a scalar process W_t whose distribution of the first hitting time must be matched. One interesting open question is to prove existence of the solution to the Generalized Shiryaev's problem when τ has a continuous infinitely divisible distribution. Another interesting generalization that requires much attention is the Generalized Shiryaev's problem for a vector-valued process W_t . Such generalization will prove extremely useful in application settings as well as providing an interesting math-

ematical playground.

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